A SELECTION THEOREM

BY

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ABSTRACT

It is shown that under certain assumptions the existence of a continuous selection on every compact subset of a metric space implies the existence of a continuous selection on the whole space. An application of this result to a problem concerning the extension of compact operators is given.

Let ϕ be a map from a metric space M to the set 2^B consisting of all nonempty subsets of a Banach space B. We say that ϕ admits a continuous selection if there exists a continuous function f from M to B (in the norm topology) such that $f(m) \in \phi(m)$ for every $m \in M$. This notion was investigated by E. Michael in a long series of papers. We shall here use one of Michael's results to prove the following

THEOREM. *Let M be a metric space and let B be a Banach space. Let ϕ be a map from M to 2^B such that $\phi(m)$ is a closed, convex and separable subset of B for every $m \in M$. Assume that for every countable compact subset K of M the restriction of ϕ to K admits a continuous selection. Then ϕ admits a continuous selection.*

NOTATIONS. Let Y be a metric space, let $y_0 \in Y$ and let $r \geqq 0$. The cell $\{y;\, y \in Y,\, d(y, y_0) \leqq r\}$ is denoted by $S_Y(y_0, r)$. $\bar{A}$ denotes the closure of the set A.

Proof of the Theorem. Let $m \in M$. For every countable compact subset K of M containing m, put

$$\psi(m,K) = \{x;\ \text{there is a continuous function } f \text{ from } K \text{ to } B \text{ with } f(m) = x \text{ and } f(k) \in \phi(k) \text{ for all } k \in K.\}$$

By our assumptions $\psi(m,K) \neq \emptyset$ for every K containing m. It is easily verified that $\psi(m,K)$ is a convex subset of $\phi(m)$. Let $\{K_i\}_{i=1}^{\infty}$ be a sequence of countable compact subsets of M such that $m \in \bigcap_{i=1}^{\infty} K_i$. Let

$$K = \bigcup_{i=1}^{\infty} (K_i \cap S_M(m, 1/i)).$$

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K is a countable compact set containing m and we claim that

$$\psi(m,K) \subset \bigcap_{i=1}^{\infty} \psi(m,K_i). \tag{1}$$

Indeed, let f be a continuous function from K to B such that $f(k) \in \phi(k)$, for all $k \in K$. Put $x = f(m)$ and fix an i, $1 \leqq i < \infty$. We shall show that $x \in \psi(m, K_i)$. By our assumptions there is a continuous function g from K_i to B such that $g(k) \in \phi(k)$ for all $k \in K_i$. Let $\alpha(k)$ be a continuous real-valued function on K_i such that $0 \leqq \alpha(k) \leqq 1$ for every k, $\alpha(m) = 1$ and $\alpha(k) = 0$ if $k \notin S_M(m, 1/i)$. Let F be the function from K_i to B defined by

$$F(k) = \begin{cases} \alpha(k)f(k) + (1-\alpha(k))\,g(k) & \text{if } k \in K_i \cap S_M(m,1/i) \\ g(k) & \text{if } k \in K_i \text{ and } k \notin S_M(m,1/i). \end{cases}$$

It is clear that F is continuous, $F(m) = x$ and $F(k) \in \phi(k)$ for every $k \in K_i$. This proves (1).

Since $\phi(m)$ is separable, it follows from (1) that

$$\psi(m) = \bigcap_{m \in K} \overline{\psi(m,K)} \neq \emptyset.$$

(The intersection is taken over all countable compact K which contain m.)

We show next that the mapping $m \to \psi(m)$ from M to 2^B is lower semi-continuous in the terminology of Michael [2]. That is, we have to show that, for every $m \in M$, every $x \in \psi(m)$ and every sequence $\{m_i\}_{i=1}^{\infty} \subset M$ with $m_i \to m$, there are $x_i \in \psi(m_i)$ $(i = 1, 2, \cdots)$ satisfying $x_i \to x$. Suppose this were false. Then there are an $m \in M$, an $x \in \psi(m)$, a sequence $m_i \to m$ and an $\varepsilon > 0$ such that $\psi(m_i) \cap S_B(x,\varepsilon) = \emptyset$ for every i. From the fact that $\phi(m_i) \cap S_B(x,\varepsilon)$ is separable and from (1) we infer that for every i there is a countable compact set K_i, with $m_i \in K_i$ such that

$$\psi(m_i, K_i) \cap S_B(x,\varepsilon) = \emptyset. \tag{2}$$

The arguments used above show also that if K_i' and K_i'' are two countable compact sets such that $m_i \in K_i' \cap K_i''$ and such that there is a neighborhood G of m_i satisfying $G \cap K_i' = G \cap K_i''$ then $\psi(m_i, K_i') = \psi(m_i, K_i'')$. Hence it is possible to choose the sets K_i which satisfy (2) so that also $K_i \subset S_B(m_i, 1/i)$. Let $K = \{m\} \cup \bigcup_{i=1}^{\infty} K_i$. K is a countable compact set and hence there is a continuous function f from K to B such that $\|f(m) - x\| \leqq \varepsilon/2$ and $f(k) \in \phi(k)$, $k \in K$. Hence $\|f(m_i) - x\| \leqq \varepsilon$ for large enough i and this contradicts (2).

We can now apply Theorem 3.2″ of [2] to obtain a continuous function f from M to B with $f(m) \in \psi(m) \subset \phi(m)$ for every $m \in M$. This concludes the proof of the theorem.

The theorem will hold also in more general situations. For example the proof we gave above remains valid if B is a general Fréchet space (since Theorem 3.2″ of [2] is valid for such B). Using other selecti on theorems of Michael it is possible to obtain results similar to the theorem proved here. We shall not go here into the details of the possible generalizations. We give only a few examples which show the role of some of the assumptions appearing in the statement of the theorem.

EXAMPLE 1. Let $B = M$ be the real line R. For every $r \in R$ let $\phi(r)$ be the set of all the integers which are greater than r. For every compact subset K of R the restriction of ϕ to K admits a continuous selection, but ϕ itself does not admit a continuous selection. Here $\phi(r)$ is closed and separable for every r but not convex.

EXAMPLE 2. Let I be an uncountable set and let $B = M = l_1(I)$ (the space of all real-valued functions α defines on I and satisfying $\Sigma_i |\alpha(i)| = \|\alpha\| < \infty$). For $\alpha \in l_1(I)$ denote by $s(\alpha)$ the set $\{i;\ i \in I,\ \alpha(i) \neq 0\}$, and put

$$\phi(\alpha) = \{\beta;\ \beta \in l_1(I),\ s(\beta) \cap s(\alpha) = \emptyset\ ,\ \|\beta\| = 1, \quad \beta \geqq 0\}.$$

Every $\phi(\alpha)$ is closed and convex. If X is a separable subspace of M then there is an $i_0 \in I$ such that $\alpha(i_0) = 0$ for every $\alpha \in X$. Let β_0 be the elementt of $l_1(I)$ defined by $\beta_0(i_0) = 1$ and $\beta(i) = 0$ if $i \neq i_0$. $\beta_0 \in \phi(\alpha)$ for every $\alpha \in X$ and hence the restriction of ϕ to X admits a continuous selection (the function identically equal to β_0). However ϕ itself does not admit a continuous selection. To see this suppose that $f(\alpha) \in \phi(\alpha)$ for every $\alpha \in l_1(I)$, and let $\alpha_0 = f(0)$. Then $\alpha_0/n \to 0$ but $\|f(\alpha_0/n) - f(0)\| = 2$ for every n (since $\|f(0)\| = \|f(\alpha_0/n)\| = 1$ and $s(f(0)) \cap s(f(\alpha_0/n)) = \emptyset$).

EXAMPLE 3. Let M be the subset of R^2 (the Euclidean plane) consisting of the points $(1/n, 1/m)$, $1 \leqq n, m < \infty$, the points $(1/n, 0)$ $1 \leqq n < \infty$ and the origin $(0,0)$. Let $B = R^2$. Let $\phi(1/n,\ 1/m)$, $1 \leqq n,\ m < \infty$, be the line segment joining the point $(0, 1/n)$ with the point $(1, 0)$, let $\phi(1/n, 0)$, $1 \leqq n < \infty$, be the line segment joining the point $(0, 0)$ with the point $(1, 0)$ and let $\phi(0, 0) = \{(0,0)\}$. $\phi(p)$ is a compact convex set for every $p \in M$. It is easily checked that the res.riction of ϕ to every convergent sequence K in M admits a continuous selection but ϕ itself does not admit a continuous selection.

We shall apply now the theorem which was proved above to a question which was considered in Chapter VII of [1]. Let $X \subset Y$ be a Banach spaces. We say that there is a continuous norm-preserving extension (C. N. P. E) map from X^* to Y^* if there is a continuous function f from X^* to Y^* (in the norm topologies) such that for every $x^* \in X^*$, we have $\|f(x^*)\| = \|x^*\|$ and the restriction of $f(x^*)$ to X is equal to x^*. The following corollary is an immediate consequence of the selection theorem proved here and the familiar representation of compact operators with range in a $C(K)$ space (cf. also [1, Lemma 7.2]).

COROLLARY. *Let $X \subset Y$ be Banach spaces and assume that for every $x^* \in X^*$ the set of its norm-preserving extensions to Y is separable (in the norm topology of Y^*). Then the following statements are equivalent.*

(i) *There is a* C. N. P. E. *map from X^* to Y^*.*

(ii) *For every compact Hausdorff space K, every compact linear operator from X to $C(K)$ has a compact and norm-preserving extension from Y to $C(K)$.*

(iii) *For every countable compact metric space K every compact linear operator from X to $C(K)$ has a compact norm preserving extension from Y to $C(K)$.*

We do not know whether the corollary will still hold if we drop the separability assumption appearing in its statement.

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Remarks added in proof. (1) It is easily seen from the proof of the theorem that it will still hold if we replace the assumption that $\phi(m)$ is separable for every m by the assumption that $\phi(m)$ is Lindelof in the w topology for every m. The same is true for the corollary. In particular the corollary holds whenever Y/X is reflexive even if we drop the separability assumption.

(2) In the selection problem appearing in the corollary the sets $\phi(m), m \in M$, are mutually disjoint (to different functionals on X correspond disjoint sets of extensions to Y). In example 2 the sets $\phi(\alpha)$ are very far from being disjoint (any countable number of them intersect). One may be inclined to think that the separability assumption in the theorem may be removed or relaxed if we make the additional assumption that the sets $\phi(m)$ are mutually disjoint. However, as pointed out by E. Michael, this is not the case—in fact every selection problem can be reformulated as a selection problem with disjoint sets $\phi(m)$ Let $\phi : M \to 2^B$ with M a metric space and B a Banach space. Let B_0 be a Banach space containing M (isometrically). Define $\psi : M \to 2^{B+B_0}$ by putting $\psi(m) = \{y; y = (x, m),\ x \in \phi(m)\}$. Then ψ admits a continuous selection iff ϕ admits a continuous selection and the sets $\psi(m)$ are mutually disjoint,

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